NGUYEN HIEN HIGHSCHOOL

ADVANCED MATHEMATICS

PART I

LINEAR SYSTEMS

LINEAR SYSTEMS

UNITE 1 : LINEAR SYSTEMS IN THREE VARIABLE

I./ What is a linear system in three variables ?

A linear equations in three variable x, y, z is represented as ax+by+cz =d (*) where x,y,z are the unknown variables, the given numbers a,b,c,d are called coefficients.

Note that a,b,c are never simultaneuosly equal to 0.

An ordered triple $(x_0;y_0;z_0)$ satisfied the equation (*) is called a solution to the equation (*).

A linear system of three equations in three variables x,y,z is represented as :

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} \quad (**)$$

where x,y,z are the unknown variables, the given numbers a_i,b_i,c_i,d_i (i=1,2,3) are called coefficients.

Note that a_i,b_i,c_i (i=1,2,3) are never simultaneuosly equal to 0.

An ordered triple $(x_0;y_0;z_0)$ satisfied the equation (*) is called a solution to the equation (**).

Solving a linear system of three equations in three variables means finding all solutions to the system.

Frequently, a linear system of three equations in three variables is called a linear system in three variables for short.

<u>Example 1</u>: Given the following system, consider wherether each of them is a linear system in three vairables or not. After that, verify if the other triple (1;2;2) or (-1;2;3) is a solution to the system.

$$\begin{cases} 2x - 3y + 4z = 4 \\ -x + 2y + z = 8 \\ 3x + 4y - z = 2 \end{cases} \quad (a) \qquad \begin{cases} 3x - 2y^2 + 4z = 6 \\ 4x - 5y + 2z = -3 \\ x + 3y - z = -1 \end{cases} \quad (b)$$

a./ This is a linear system in three variables x,y,z.
To check if the ordered triple (1;2;2) is a solution to (a), substitute 1,2,2 for x,y,z respectively :

2-6+8 =4 (true)

-1+4+2 =8 (false)

Hence, the ordered triple (1;2;2) is not a solution to the system (a)

Similarly, substitute -1;2;3 for x;y;z respectively :

-2-6+12=4 (true)

1+4+3=8 (true)

-3 +8-3 =2 (true)

Therefore, the ordered triple (-1;2;2) is a solution to the system (a) .

b ./ This is not a linear system since ther exists y^2 .

II./ The Gaussian Elimination method

Any linear system in three variable x,y,z can be converted into a triangular form. An example for linear system in triangular form is as below

$$\begin{cases} x - 3y + 2z = 4 \\ \quad 3y - 9z = 5 \\ \qquad\quad z = 2 \end{cases}$$

Gaussian Elimination method can be used to solve a system of three linear equations in three variables.

Step1: Convert the system into an equivalent triangular form.

Step 2: Solve for one of variables, then substitute to the orther equations to solve for the remaining variables.

Example 2: Solve by Gaussian Elimination method :

$$\begin{cases} 6x + 2y + z = 380 \quad (1) \\ 3x + y + 2z = 340 \quad (2) \\ x + 4y + 5z = 620 \quad (3) \end{cases}$$

Step 1: Convert the system into an equivalent triangular form.

Multiply the equation (2) by 2 and multiply the equation (3) by 6:

$$\begin{cases} 6x + 2y + z = 380 \quad (1.1) \\ 6x + 2y + 4z = 680 \quad (2.1) \\ 6x + 24y + 30z = 3720 \quad (3.1) \end{cases}$$

Subtract the equation (1.1) from (2.1) and subtract the equation (2.1) from (3.1)

$$\begin{cases} 3y = 300 \quad (1.2) \\ 22y + 26z = 3040 \quad (2.2) \\ x + 4y + 5z = 620 \quad (3.2) \end{cases}$$

This is in triangular form.

Step 2: Find solution to the system

From (1.2), we obtain z =100

Next, we substitute z =100 into the equation (2.2) and solve for y

$$22y + 26x100 = 3040$$
$$22y = 440$$
$$y = 20$$

Finally, substitute z= 100 and y =20 into the equation (3.2), then solve for x

$$x + 4x20 + 5x100 = 620$$
$$x = 40$$

Therefor, the solution is the ordered triple (40;20;100).

Note : A linear system in three variables might have only one solution, or infinitely, many solutions.

III./ Solving linear system in three variables using a caculator

Instructions for Casio fx-580 -vn , fx -570-vn

Casio fx-570 –vn	Casio fx-580 -vn
Step 1 : Open the three variable linear system solver mode MODE 5 2	*Step 1*: Open the three variable linear system solver mode MENU 9 1 3
Step 2: Type in the coefficients of equations	*Step 2* : Type in the coefficients of equations
Step 3 : Press = to find the solution	*Step 3* : Press = to find the solution

Example 3 : Use a calculator to solve the following systems

$$a)\begin{cases}3x-y+z=3\\x-y+z=2\\x+2z=1\end{cases}\qquad b)\begin{cases}2x+2y-z=-1\\x+4y+z=-8\\x-2y-2z=7\end{cases}$$

$$c)\begin{cases}x-2y+3z=9\\2x+3y-z=4\\x+5y-4z=2\end{cases}$$

(a) The solution is $(\frac{1}{2};-\frac{2}{3};\frac{5}{6})$
(b) There are infinitely many solutions.
(c) There are no solution.

Exercises

1./ Which one of the followings is a linear system in three variables ? Verify if the ordered triple (-1;2;1) or (-1,5 ; 0,25 ; -1,25) is a solution to the linear system.

$$a)\begin{cases}3x-2y+z=-6\\-2x+y+3z=7\\4x-y+7z=1\end{cases}\qquad b)\begin{cases}5x-2y+3z=4\\3x+2yz-z=7\\x-3y+2z=1\end{cases}\qquad c)\begin{cases}2x-4y-3z=-0,25\\3x+8y-4z=2,5\\2x+3y-2z=0,25\end{cases}$$

2./ Solve these systems by Gaussian Elimination method, then use a calculator to verify the results:

$$a)\begin{cases}3x-y+z=4\\x-y+z=2\\y+2z=8\end{cases}\qquad b)\begin{cases}x+2y-z=1\\x+4y+z=2\\x-2y-3z=-1\end{cases}\qquad c)\begin{cases}-x+5y+z=-4\\x+y=3\\y+2z=3\end{cases}$$

$$d)\begin{cases}2x-4y+6z=0\\x-2y+3z=1\\x+z=-1\end{cases}\qquad e)\begin{cases}x+2z=5\\2x-y+z=-1\\3x+2y=-7\end{cases}\qquad f)\begin{cases}x-2y+3z=1\\-2x+4y-6z=-2\\x+y+z=2\end{cases}$$

3./ Find the equation in general form $y=ax^2+bx+c$ ($a \neq 0$) of a parabola given that:
a./ The Parabola has the line of symmetry x=1 and (P) passes through two points A(1;-4) and B(2;-3).
b./ The Parabola has the vertex I (0,5; 0,75) and passes through the point M(-1;3).

4./ Long, Minh, and Nam went to a cafeteria. Long paid 90 000 VND for a bottle of apple juice, a banana, and two sandwiches. Minh spent 140 000 VND for a bottle of apple juice, two bananas, and three sandwiches. Nam bought a bottle of apple juice and three sandwiches at a cost of 50 000 VND. Let l,m,n be the price of a bottle of apple juice, a banana, and a sandwich, respectively.

a./ Express three equations in the variables l,m,n.

b./ Find the values of l,m,n.

UNIT 2 : APPLICATIONS OF LINEAR SYSTEMS OF EQUATIONS

I./ Steps to solve word problems applying linear systems in three variables

A word problem requires you to find an answer from the facts of problem. Some word problems require the use of linear systems of equations. Here are steps to solve word problems applying linear systems in three variable:

Step 1: Build a linear system:

- Assign variables to represent the unknown quantities.
- Find all contraints for the variables.
- Write linear equations based on the given information

Step 2: Sovle the linear system.
Step 3: Verify and interpret the solutions in the context of the real-world problem.

Example 1:

Anh and his friends bought pastels, brushes, and canvases at an art supply store. The table shows the number of each item bought and the amount each person spent. Identify the price of each item.

	Pastels	Brushes	Canvases	Total (VND)
Anh	3	4	5	440000
Lam	2	5	7	460000
Khoi	4	3	6	480000

Solution

Let p,b,c (thousand VND) be the price of a pastel, brush, a canvas, respectively p,b,c ≥ 0.
From the first row of the table
$3p + 4b + 5c = 440$.
From the second row of the table
$2p + 5b + 7c = 460$.
From the third row of the table
$4p + 3b + 6c = 480$.
Hence, we obtain the system of three- variable linear equations as follow :

$$\begin{cases} 3p + 4b + 5c = 440 \\ 2p + 5b + 7c = 460 \\ 4p + 3b + 6c = 480 \end{cases}$$

By using a calculator, the solution for (p;b;c) is (60;40;20).

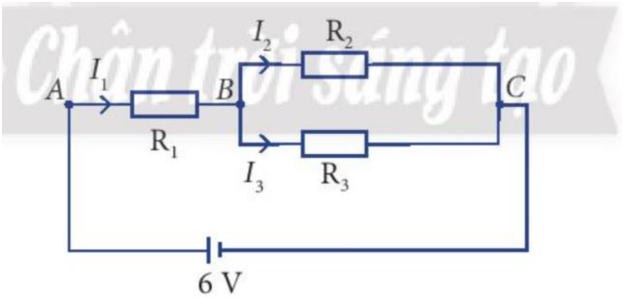
Therefore, a pastel costs 60 000 VND, a brush costs 40 000 VND, and a canvas costs 20 000 VND .

II./ Application in Physics, Chemisty, and Biology.

1./ Solving electrical circuit problems

Examble 2 :

In an electric circuit, the resistors $R_1 = 6\ \Omega$, $R_2 = 4\ \Omega$, and $R_3 = 3\ \Omega$ are connected as in figure . What are the values of the currents I_1, I_2, I_3 ?



Solution

Since the total current flowing into B is equal to the total current flowing out of B,

$I_1 = I_2 + I_3$

The voltage between A and C can be represented as either $U_{BC} = I_2 . R_2$ or $U_{BC} = I_3 . R_3$, so

$4I_2 = 3I_3$

The voltage between A and C can be represented as $U_{AC} = I_1 . R_1 + I_3 . R_3$.

It is given that $U_{AC} = 6$, hence,

$6I_1 + 3I_3 = 6$

Therefore, the system of linear equations is

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 4I_2 - 3I_3 = 0 \\ 2I_1 + I_3 = 2 \end{cases}$$

Solving by using a calculator, the current through each resistor is $I_1 = \frac{7}{9}A;\ I_2 = \frac{1}{3}A\ :I_3 = \frac{4}{9}A$.

2./ Balancing chemical equations

Example 3:

Balancing the chemical equation for the combustion of propane:

$$C_3H_8 + O_2 \rightarrow CO_2 + H_2O$$

Solution

Let n_1, n_2,n_3,n_4 be positive integers such that :

$$n_1.\ C_3H_8 + n_2.\ O_2 \rightarrow n_3.\ CO_2 + n_4.\ H_2O$$

Balancing the number of atoms of Hydrogen on both sides gives $8n_1 = 2n_4$

Balancing the number of atoms of Carbon on both sides gives $3n_1 = n_3$

Balancing the number of atoms of Oxygen on both sides gives $2n_2 = 2n_3 + n_4$

We obtain the following linear system :

$$\begin{cases} 8n_1 = 2n_4 \\ 3n_1 = n_3 \\ 2n_2 = 2n_3 + n_4 \end{cases}$$

The system has infinitely many solutions.

Since n_1, n_2,n_3,n_4 are positive integers, let's choose $n_4 = 4$, then

$$\begin{cases} n_1 = \frac{1}{4}n_4 = \frac{1}{4}.4 = 1 \\ n_3 = 3n_1 = 3.1 = 3 \\ n_2 = \frac{2n_3 + n_4}{2} = 5 \end{cases}$$

Hence, the chemical equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

3./ Solving cell division problems

Example 4:

Three cell A,B and C after a number of minotic divisions made 88 daughter cells. Daughter cells produced by cell B twice as many as those of cell A. The number of minotic divisions of cell C is 2 more than that of cell B. Given that a cell after one minotic division creates two daughter cells, identify each cell's number of minotic divisions.

Solution

Let a,b,c respectively represent the number of divisions of cells A, B, C (a,b,c > 0)
The total number 88 of daughter cells gives $2^a + 2^b + 2^c = 88$
Cell B created twice as many daughters as cell A gives $2^b = 2.\,2^a$.
The number of minotic divisions of cell C being 2 more than that of cell means c=b+2
We obtain the system

$$\begin{cases} 2^a + 2^b + 2^c = 88 \\ 2^b = 2.\,2^a \\ c = b + 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2^a + 2^b + 2^c = 88 \\ -2.2^a + 2^b = 0 \\ -4.2^b + 2^c = 0 \end{cases}$$

Denote $x=2^a$, $y=2^b$, $z= 2^c$, the system becomes linear in three variables x, y, z :

$$\begin{cases} x + y + z = 88 \\ 2x + y = 0 \\ -4y + z = 0 \end{cases}$$

Solving by using a calculator, the solution for (a,b,c) is (8,16,64).
Therefore,

$$\begin{cases} 2^a = 8 \\ 2^b = 16 \\ 2^c = 64 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 3 \\ b = 4 \\ c = 6 \end{cases}$$

Hence, the number of minotic divisions of cells A, B, C is 3, 4 , 6 .

III./ Application in Economics

Example 5.

John won 86 000 dollars in the local lottery. He decided to invest the whole prize in three different investment funds.

• One dollar a REIT pays 6 percent annual simple interest. The others, a C.D and a Bond

fund pays 2 percent and 3 percent respectively.
• The total in annual simple interest from the three investments was 3900 dollars.
• The annual simple interest earned from the REIT was 900 dollars less than three times
the amount earned from the Bond fund and C.D.
Find how much was invested in each fund.

Solution

Let r, c, b be the amount invested in the REIT, C.D, and Bond fund, respectively, (r, c, b>0).
The whole amount 86 000 dollars means

$$r + c + b = 86000.$$

The simple interests of 6, 2, 3 respectively, and the total 3900 dollars in annual simple interest
gives

$$0.06r + 0.02c + 0.03b = 3900.$$

The annual simple interest earned from the REIT was 900 dollars less than three times the
amount earned from the Bond fund and C.D, which can be described mathematically as

$$0.06r = 3(0.02c +0.03b) - 900.$$

The following system of linear equations is obtained:

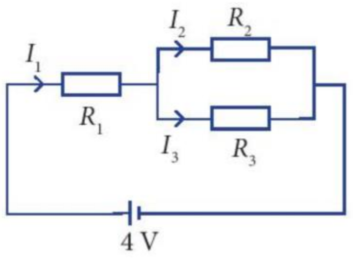
$$\begin{cases} r + c + b = 86000 \\ 0.06r + 0.02c + 0.03b = 3900 \\ 0.06r - 0.06c - 0.09b = -900. \end{cases}$$

By using a calculator, the solution for (r, c, b) is (38 000, 45 000, 3 000).
Therefore, John invested 38 000 dollars in REIT, 45 000 dollars in CD, and 3000 dollars in Bond fund.

EXERCISES

1. Costumes are needed for the school play. Boy's costumes require 8 yards of fabric, 5 yards of ribbon,and 4 packets of sequins. Girl's costumes require 6 yards of fabric, 9 yards of ribbon, and 5 packets of sequins. Fabric costs 6 dollars per yard, ribbon costs 5 dollars per yard, and sequins cost 5 dollars per packet. Find the cost of each costume.

2. Given an electric circuit, the resistors $R_1 = 4\Omega$, $R_2 = 4\ \Omega$, and $R_3 = 8\ \Omega$. What are the values of the currents I_1, I_2, I_3 ?.



3. Balance the chemical equation:

$$CH_4 + O_2 \xrightarrow{t^o} CO_2 + H_2O.$$

4. To study the effect of a mixture of three types of vitamins, a biologist provides each rabbit in a laboratory exactly 15 mg thiamine (B1), 40 mg riboflavin (B2) and 10 mg niacin (B3) a day. There are three types of foods with the amount of vitamins given as below:

	Food type I	Food type II	Food type III
B1	3	2	2
B2	7	5	7
B3	2	2	1

(milligram(s) vitamins in 100 g food)

How much gram of each type of foods should a rabbit be provided a day?

5. The Soft-flow Yogurt Company makes three yogurt blends: Lime Orange, using 2 quarts of lime yogurt and 2 quarts of orange yogurt per gallon; Lime Lemon, using 3 quarts of lime yogurt and 1 quart of lemon yogurt per gallon; and Orange Lemon, using 3 quarts of orange yogurt and 1 quart of lemon yogurt per gallon. Each day day company has 800 quarts of lime yogurt, 650 quarts of orange yogurt, and 350 quarts of lemon yogurt available. How many gallons of each blend should the company make each day if it wants to use up the supplies?

6. Three types of cells A, B,C experienced 3,4,7 mitotic divisions and the total number of daughter cells was 480. Before mitosis, the number of cells B was the sum of those A and C. After mitosis, the sum of daughter cells of A and C was five times as many as B. Find the number of parent cells of each type.

7. The bar graph (figure) compares the body composition of a 160-pound adult male and a 135-pound adult female.

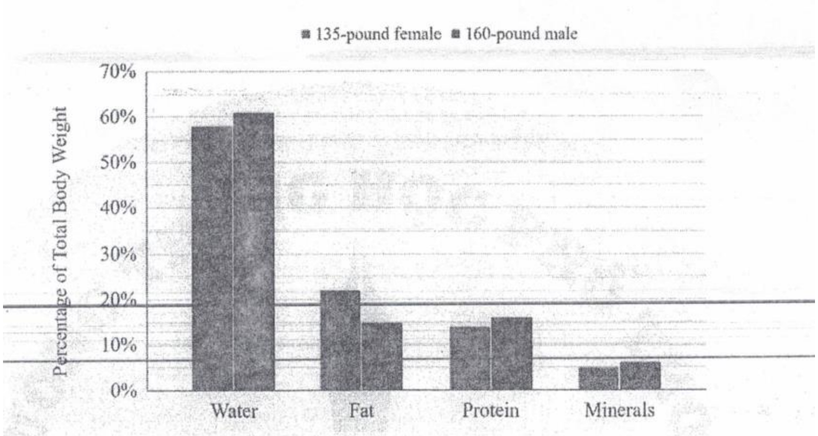


Figure 3 Body composition of an Adult Male and Female

(a) This problem refers to the body composition of a 135-pound adult female, shown by the graph. Ninety-five percent of her total body weight consists of water, fat, and protein. The difference between the percentage of body weight consisting of water and fat is 35 percent. The difference between the percentage of body weight consisting of fat and protein is 9 percent. Find the percentage of total body weight consisting of water, of fat, and of protein.

(b) This problem refers to the body composition of a 160-pound adult male, shown by the graph. Ninety-four percent of his total body weight consists of water, fat, and protein. The difference between the percentage of body weight consisting of water and fat is 47 percent. The difference between the percentage of body weight consisting of protein and fat is 2 percent. Find the percentage of total body weight consisting of water, of fat, and of protein.

PRACTICE

1./ Which one of the followings is a linear system in three variables? Verify if the ordered triple (-1,1,0) or (0,5;-0,5; -1) is a solution to the linear system.

a) $\begin{cases} 2x - y + z = -1 \\ -x + 2y = 1 \\ 3y - 2z = -2; \end{cases}$ b) $\begin{cases} 4x - 2y + z = 2 \\ 8x + 3z = 1 \\ -6y + 2z = 1; \end{cases}$ c) $\begin{cases} 3x - 2y + zx = 2 \\ xy - y + 2z = 1 \\ x + 2y - 3yz = -2. \end{cases}$

2. Solve each of the following systems by Gaussian Elimination method, then use a calculator to verify the results.

a) $\begin{cases} x - 2y + z = 3 \\ -y + z = 2 \\ y + 2z = 1; \end{cases}$ b) $\begin{cases} 3x - 2y - 4z = 3 \\ 4x + 6y - z = 17 \\ x + 2y = 5; \end{cases}$ c) $\begin{cases} x + y + z = 1 \\ 3x - y - z = 4 \\ x + 5y + 5z = -1. \end{cases}$

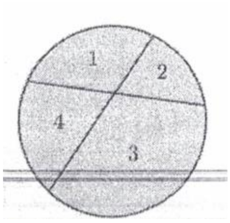
3. Find the equation of a parabola (P): $y = ax^2+bx+c$ given that:
(a) the x-intercepts of the parabola are x = -2 and x = 1, and the curve passes through point M(-1, 3);
(b) the y-intercept of the parabola is y= -2 and the minimum value of the curve is - 4 at x = 2.

4. A boat traveled 45 miles downstream and back. The trip downstream took 3 hours and the trip back took 5 hours. Find the average speed of the boat in still water and the speed of the current.

5. The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

6. Cutting a circular pizza using n straight cuts, the maximum number of pieces that can be made is given by $P(n)=an^2+bn+c$ where a, b, c are constants. For example,

$P(2) = 4a + 2b + c = 4$



means that we can make a maximum of 4 pieces with two cuts (figure).
(a) Write three linear equations involving a, b, and c.
(b) Solve the linear system and determine P(n).
(c) Hence find the maximum number of pieces that can be made using 12 cuts.

7. A ball was thrown straight up from a rooftop. On the way down, it missed the rooftop and hit the ground. A mathematical model can be used to describe the height of the ball from the ground, after t seconds. Consider the following data:

t seconds after the ball is thrown	h feet height above the ground
1	224
2	176
3	104

(a) Find the quadratic function $h = at^2 + bt+c$ which describe the height of the ball after t seconds.
(b) What is the height of the ball after 5 seconds?

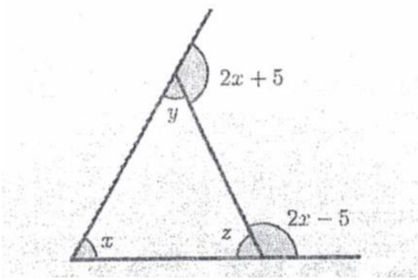
8. Three sets of triathlon teams are competing in separate triathlon legs, a swim, a bike, and a run, for their training. The speeds of the participants and the total distance covered by the team are in the table below.

Team	Swim	Bike	Run	Total Distance Covered by Team
1	2 mph	17 mph	7.5 mph	47.75 miles
2	1.5 mph	20 mph	7 mph	52.375 miles
3	1.25 mph	18 mph	8 mph	49.5625 miles

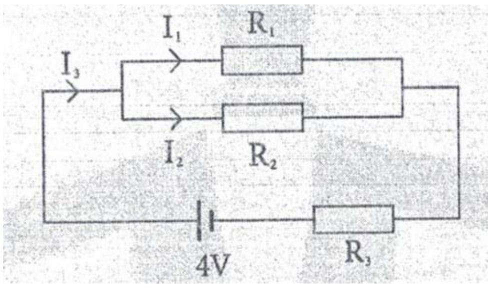
How much time is each leg of the training race?

9. A football team scored a total of 50 points in one game. They scored 14 times. The scoring was made up of touchdowns (6 points), PATs (1 point each), and field goals (3 points each). They had three more touchdowns than field goals. How many of each type of score did the team have?

10. In the following triangle, the measurements of the angles are represented in three variables x, y, and z. Find the measurement of each interior angle.



11. Given an electric circuit below, the values of resistors are $R_3= R_2=R_1=5\,\Omega$. Find the currents I_1, I_2,I_3.



12. A person invested 6700 dollars for one year, part at 8 percent, part at 10 percent, and the remainder at 12 percent. The total annual income from these investments was 716 dollars. The amount of money invested at 12 percent was 300 dollars more than the amount invested at 8 percent and 10 percent combined. Find the amount invested at each rate.

13. Three neighbors make a fruit salad for a summer picnic. One person uses three pounds of strawberries plus five pounds of grapes plus one pound of melon at a cost of 20 dollars. A second person uses three pounds of strawberries plus two pounds of grapes plus two pounds of melon at a cost of 21 dollars. The last neighbor uses four pounds of strawberries plus three pounds of grapes plus three pounds of melon at a cost of 30 dollars. How much does each fruit cost?

14. A chemist has three different acid solutions. The first acid solution contains 10 percent acid, the second contains 40 percent acid and the third 60 percent acid. The chemist wants to use all three solutions to obtain a mixture of 100 liters containing 45 percent acid. The supply of 40 percent solution must be used. How many liters of each solution should be used?

15. Mariacopa' Success scholarship fund receives a gift of 200 000 dollars. The money is invested in stocks, bonds, and CDs. CDs pay 3.1 percent interest, bonds pay 4.25 percent interest, and stocks pay 8.2 percent interest. Maricopa Success invests 25 000 dollars more in bonds than in stocks. If the annual income from the investment is 9925 dollars, how much was invested in each account?

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